Appendix 1.3

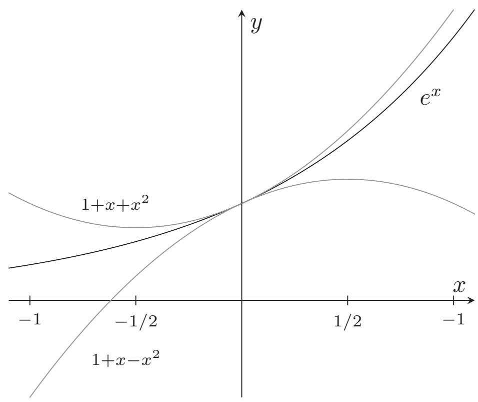
1. **Bounded convergent sequences.** If $\{a_n\}_{n\geq 1}$ is a sequence that converges to ℓ as $n\to\infty$ and the a_n are bounded for all $n\geq 1$, i.e. $|a_n|\leq B$ for some $B>0$, then $|\ell|\leq B$.

   Proof by contradiction. Assume $|\ell|>B$. Choose $\varepsilon=(|\ell|-B)/2>0$ in the definition of $\lim_{n\to\infty}a_n=\ell$ to find $N\geq 1$ such that if $n\geq N$ then $|a_n-\ell|<\varepsilon$. Pick any $n_0\geq N$, then, by a form of the triangle inequality

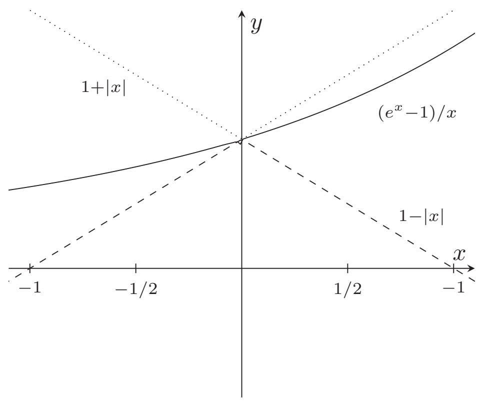
$$\begin{aligned}|a_{n_0}| &= |\ell+a_{n_0}-\ell|\geq|\ell|-|a_{n_0}-\ell|\\ &> |\ell|-\varepsilon=|\ell|-\frac{|\ell|-B}{2}\\ &= \frac{|\ell|+B}{2}>\frac{2B}{2}=B,\end{aligned}$$

   making use of the assumption $|\ell|>B$. Thus $|a_{n_0}|>B$, contradicting the fact that $|a_n|\leq B$ for all $n\geq 1$. Hence assumption is false, i.e. $|\ell|\leq B$.

2. **Exponential Function: graphs** The graphs for the results of Theorem 2 are



and



Though they suffice for the applications of the Sandwich Rule the lower bounds seem particularly poor. It would appear from the graphs that we should have

$$e^x > 1 + x \quad \text{and} \quad \frac{e^x - 1}{x} > 1 + \frac{x}{2}.$$

for all real x. Can you prove these, especially for negative x?

3. **Exponential Function** In the lectures we proved

$$|e^x - 1 - x| \leq |x|^2$$

for $|x| < 1/2$. The method of proof can be extended.

Lemma *Prove that for all* $k \geq 1$,

$$\left| e^x - \sum_{r=0}^{k} \frac{x^r}{r!} \right| \leq \frac{2\,|x|^{k+1}}{(k+1)!} \tag{5}$$

for $|x| < 1/2$.

Solution Start from the definition of an infinite series as the limit of the sequence of partial sums, so

$$e^x - \sum_{r=0}^{k} \frac{x^r}{r!} = \lim_{N\to\infty} \sum_{r=k+1}^{N} \frac{x^r}{r!} = x^{k+1} \lim_{N\to\infty} \sum_{j=0}^{N-k-1} \frac{x^j}{(j+k+1)!}. \tag{6}$$

Then, by the triangle inequality, (applicable since we have a **finite** sum),

$$\left|\sum_{j=0}^{N-k-1} \frac{x^j}{(j+k+1)!}\right| \le \sum_{j=0}^{N-k-1} \frac{|x|^j}{(j+k+1)!} \le \frac{1}{(k+1)!}\sum_{j=0}^{N-k-1} |x|^j$$

since $(j+k+1)! \ge (k+1)!$ for all $j \ge 0$,

$$= \frac{1}{(k+1)!}\left(\frac{1-|x|^{N-k}}{1-|x|}\right),$$

on summing the Geometric Series, allowable when $|x| \ne 1$. In fact we have $|x| < 1/2 < 1$, which gives the second inequality in

$$\frac{1-|x|^{N-k}}{1-|x|} \le \frac{1}{1-|x|} < \frac{1}{1-1/2} = 2.$$

Hence

$$\left|\sum_{j=0}^{N-k-1} \frac{x^j}{(j+k+1)!}\right| \le \frac{2}{(k+1)!}$$

for all $N \ge 0$. Now use the result that if a sequence $\{a_n\}$ converges and $|a_n| \le B$ for some B and all n then $|\lim_{n\to\infty} a_n| \le B$. Thus

$$\left|\lim_{N\to\infty}\sum_{j=0}^{N-k-1} \frac{x^j}{(j+k+1)!}\right| \le \frac{2}{(k+1)!}.$$

Combined with (6) gives the required result. ■

4. **Example**

$$\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \frac{1}{2}.$$

Solution Take $k=2$ in (5) and divide the resulting inequality $|e^x - 1 - x - x^2/2| \le |x^3|/3$ through by $|x|^2$ to get

$$\left|\frac{e^x-1-x-x^2/2}{x^2}\right| \le \frac{|x|}{3}.$$

for $|x| \le 1/2$. This is just shorthand for

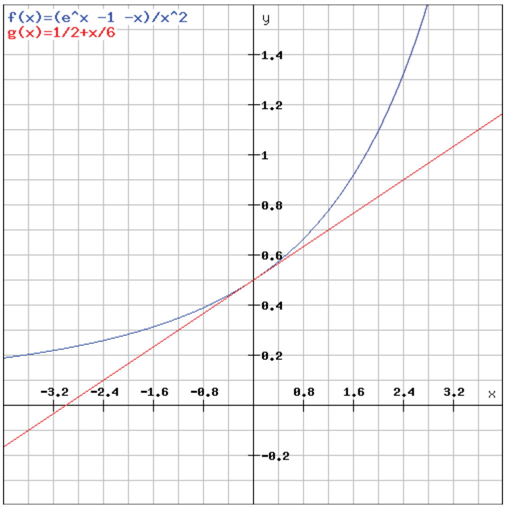
$$\frac{1}{2} - \frac{|x|}{3} \le \frac{e^x-1-x}{x^2} \le \frac{1}{2} + \frac{|x|}{3} \qquad (7)$$

for $|x| \leq 1/2$. Let $x \to 0$ when $|x| \to 0$ so, by the Sandwich Rule,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}.$$

■

From the graph the lower bound in (7) looks weak.



It looks reasonable that

$$\frac{e^x - 1 - x}{x^2} \geq \frac{1}{2} + \frac{x}{6}$$

for all $x \in \mathbb{R}$. Could you prove this?

5. **Rates of Convergence** When you have a result of the form $\lim_{x\to a} f(x) = L$ the next question might be: how 'fast' does $f(x) \to L$? How do you measure this 'speed'? Perhaps comparing $f(x) \to L$ with x tending to a, i.e. consider the ratio $(f(x) - L)/(x-a)$ and its limit,

$$\lim_{x\to a} \frac{f(x) - L}{x - a}.$$

So after $\lim_{x\to 0} e^x = 1$ in Theorem 2i. we were interested in

$$\lim_{x\to 0} \frac{e^x - 1}{x - 0} = \lim_{x\to 0} \frac{e^x - 1}{x},$$

Part ii. for Theorem 2.

We can continue for the exponential function. From $\lim_{x\to 0}\left(e^{x}-1\right)/x = 1$ we would be interested in

$$\lim_{x\to 0}\frac{\frac{e^{x}-1}{x}-1}{x-0}=\lim_{x\to 0}\frac{e^{x}-1-x}{x^{2}},$$

the subject of Example 4.

Similarly, after $\lim_{\theta\to 0}\sin\theta = 0$, we would be interested in

$$\lim_{\theta\to 0}\frac{\sin\theta-0}{\theta-0}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta},$$

the subject of Example 4. Since this limit is 1 we think of $\sin\theta$ and θ tending to 0 at the same rate.

After $\lim_{\theta\to 0}\cos\theta = 1$ we looked, in Example 5, at

$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}.$$

Since the limit is 0 we think of $\cos\theta$ tending to 1 faster than θ tending to 0. We can continue and look at

$$\lim_{\theta\to 0}\frac{\frac{\cos\theta-1}{\theta}-0}{\theta-0}=\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta^{2}}.$$

This was the subject of Example 6 where it's limit was found to be $-1/2$. We think of $\cos\theta$ tending to 1 at the same rate as θ^{2} tends to 0.